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LETTER TO THE EDITOR

Classical realisation of YBZF algebras from classical realisations of Lie algebras

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Abstract. We prove that all classical realisations of the YBZF algebras, obtained by Reshetikhin are associated with Poisson realisations of Lie algebras (corresponding to the invariance group of defining relations for YZBF algebras), which satisfy second-degree polynomial identities. Some possible extensions of this result are discussed.

The YBZF (Yang–Baxter–Zamolodchikov–Faddeev) algebras are the underlying algebraic structures to the classical or quantum integrable models [1–3]. The various concrete models are now considered as physical realisations of abstract YBZF algebras [4–8].

In the present letter, inspired by Reshetikhin's beautiful papers [4, 7], we shall consider classical realisations of YBZF algebras which are completely determined by some special classical realisations of Lie algebras studied by us in [9–13]. More precisely we shall consider the classical YBZF algebras for the local transition matrix $L_n(u)$ (L-operator) of the lattice versions of the completely integrable classical systems. In this formulation the one-site L-operator $L_n(u)$ is a function on the classical one-site phase space M (which depends also on a complex variable u called the spectral parameter) with values in $\operatorname{End} V$ where V is an auxiliary finite-dimensional linear space. The Poisson bracket $\{\ ,\ \}$, associated with the symplectic structure on the phase space M, allows [3] the introduction of the Poisson bracket $\{\overset{\otimes}{,}\}$ on the space of all smooth functions on M with values in $\operatorname{End} V$ in the following way:

$$\left\{\sum_i f_i A_i \overset{\otimes}{,} \sum_j g_j B_j\right\} = \sum_{i,j} \{f_i, g_j\} A_i \otimes B_j. \tag{1}$$

Denoting by $r(u) \in \operatorname{End}(V \otimes V)$ a given r matrix which satisfies the classical Yang–Baxter equation [6], the defining relations for the classical YBZF algebras are [4]:

$$\{L_n(u) \overset{\otimes}{,} L_{n'}(V)\} = \delta_{n,n'}[r(u-v), L_n(u) \otimes L_{n'}(v)]. \tag{2}$$

Let us now assume that a semisimple Lie group G with Lie algebra $\mathscr{G}$ acts on the manifold M and on the vectorial space V and denote by $T: \mathrm{G} \to \operatorname{End} V$ and by $(g, m) \in (\mathrm{G}, M) \to g \cdot m \in M$ these actions on V and M respectively. The invariance of the r matrix $r(u)$ is expressed by [4]:

$$(T(g) \otimes T(g)) r(u) (T(g^{-1}) \otimes T(g^{-1})) = r(u) \tag{3}$$

for all $g \in \mathrm{G}$, and the L-operator $L_n(u): M \to \operatorname{End} V$ is called equivariant if

$$L_n(u)(g \cdot m) = T(g^{-1}) L_n(u)(m) T(g). \tag{4}$$

We shall look for equivariant realisations of the YBZF algebras defined by (2).

Let $R:\mathscr{G}\to \operatorname{End} V$ denote the representation of the Lie algebra $\mathscr{G}$ corresponding to the representation T of the Lie group G. A G-invariant solution $r(u)$ of the classical Yang-Baxter equation is [3, 6]:

$$r(u)=\frac{1}{u}\sum_{i=1}^{\alpha} R(x_i)\otimes R(x^i) \tag{5}$$

where $\{x_i, i=1,2,\ldots,d(=\dim\mathscr{G})\}$ is a basis of $\mathscr{G}$, $[x_i, x_j]=\Sigma_k c_{ij}^k x_k$, and x^i is the dual basis with respect to a non-degenerate invariant bilinear form $(\,.,.)$ on $\mathscr{G}$: $(x_i, x^j)=\delta_{ij}$ for all $i, j=1, 2, \ldots, d$. The infinitesimal version of (3) is

$$[r(u), R(x)\otimes I+I\otimes R(x)]=0 \tag{6}$$

for any $x\in\mathscr{G}$, $u\in\mathbb{C}$; $I\in\operatorname{End}V$ is the identity operator on V.

Let us suppose that a Poisson bracket (PB) (i.e. classical) realisation [9-13] of the Lie algebra $\mathscr{G}$ is defined on M, i.e. that a Lie algebra homomorphism $f:\mathscr{G}\to C^\infty(M)$ (where $C^\infty(M)$ is structured as a Lie algebra by the Poisson bracket $\{.,.\}$) is given:

$$\{f_x, f_y\}(m)=f_{[x,y]}(m) \tag{7}$$

for all $x, y\in\mathscr{G}$ and any $m\in M$.

As we have shown [10-12] any pair composed from a PB realisation $f:\mathscr{G}\to C^\infty(M)$ which is equivalent, i.e.

$$f_{\operatorname{Ad}(g)x}(m)=f_x(g\cdot m) \tag{8}$$

and a representation $R:\mathscr{G}\to\operatorname{End}V$ defines an equivariant function $\mathscr{K}$ on M with values in $\operatorname{End}V$ by

$$\mathscr{K}(m)=\sum_{i=1}^{d} f_{x_i}(m)R(x^i). \tag{9}$$

The equivariance property

$$\mathscr{K}(g\cdot m)=T(g^{-1})\mathscr{K}(m)T(g) \tag{10}$$

implies that $\mathscr{K}$ maps G orbits on M into G orbits on $\operatorname{End}V$. From (6) and (9) it follows that

$$[r(u), (\mathscr{K}(m)\otimes I+I\otimes\mathscr{K}(m))^n]=0 \tag{11}$$

for all $m\in M$ and all $n\in N$. We have

$$(\mathscr{K}\otimes I+I\otimes\mathscr{K})^2=\mathscr{K}^2\otimes I+I\otimes\mathscr{K}^2+2\mathscr{K}\otimes\mathscr{K}. \tag{12}$$

Let us suppose that on a G orbit we have a polynomial identity

$$\sum_{n=0}^{2} b_n\mathscr{K}^n=0. \tag{13}$$

Then from (11) and (12) it follows that

$$[r(u), \mathscr{K}\otimes\mathscr{K}]=0. \tag{14}$$

In previous papers [11-13] we have obtained all pairs (f, R) composed from a PB realisation f and a representation R of a classical semisimple Lie algebra $\mathscr{G}$ such that the associated $\mathcal{K}$ maps (9) satisfy second-degree polynomial identities. For any such pair (f, R), there exists a classical elementary realisation of the YBZF algebra [13]:

$$L(u)(m)=\frac{1}{u+a}(uI-\mathcal{K}(m)) \tag{15}$$

where u, $a\in\mathbb{C}$.

The proof of this statement is based on (14) and the fact that

$$[x_j, x^k]=\sum_{i=1}^{d} c_{ij}^k x^i. \tag{16}$$

Indeed we have

$$\begin{aligned}(u+a)(v+a)&[r(u-v), L(u)\otimes L(v)]\\ &=(u-v)[r(u-v), \mathcal{K}\otimes I]\\ &=\left[\sum_{j=1}^{d} R(x_j)\otimes R(x^j), \sum_{k=1}^{d} f_{x_k}R(x^k)\otimes I\right]\\ &=\sum_{j,k=1}^{d} f_{x_k}[R(x_j), R(x^k)]\otimes R(x^j)\\ &=\sum_{i,j,k} c_{ij}^k f_{x_k} R(x^i)\otimes R(x^j)\end{aligned} \tag{17}$$

and by definition we have

$$(u+a)(v+a)\{L(u)(m)\underset{,}{\otimes} L(v)(m)\}=\sum_{i,j,k=1}^{d} c_{ij}^k f_{x_k}(m)R(x^i)\otimes R(x^j) \tag{18}$$

whence the equality (2) follows.

This is a general proof of Reshetikhin's results [4], in spite of an apparent difference. Indeed the $\mathcal{K}$ maps associated with the spinorial representations of the orthogonal algebras also satisfy second-degree polynomial identities [13]. This fact can also be verified directly because, in this case, Reshetikhin has given explicit formulae for the dependence of f_{x_i}, $i=1, 2, \ldots, d$, of the canonical coordinates on M.

Finally, we remark that the above considerations can be extended to the case of a polynomial identity of third degree in which case we must look for L-operators of the form

$$L(u)(m)=\sum_{n=0}^{2} h_m(u)\mathcal{K}(m)^n. \tag{19}$$

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